

Phase transition in scalar ϕ^4 -theory beyond the super daisy resummations

M. BORDAG*

University of Leipzig, Institute for Theoretical Physics
Augustusplatz 10/11, 04109 Leipzig, Germany

and

V. SKALOZUB†

Dnipropetrovsk State University, 49050 Dnipropetrovsk, Ukraine

February 7, 2008

Abstract

The temperature phase transition in scalar $\phi^4(x)$ field theory with spontaneous symmetry breaking is investigated in a partly resummed perturbative approach. The second Legendre transform is used and the resulting gap equation is considered in the extrema of the free energy functional. It is found that the phase transition is of first order in the super daisy as well as in a certain beyond super daisy resummations. There no unwanted imaginary parts in the free energy are found but a loss of the smallness of the effective expansion parameter near the phase transition temperature is found in both cases. This means an insufficiency of the resummations or a deficit of the perturbative approach.

1 Introduction

The question on the order of the phase transition when a spontaneously broken symmetry is restored by raising temperature is a topic of considerable interest. The present status of the problem in scalar ϕ^4 -theory with spontaneous symmetry breaking due to the mass term with 'wrong sign' is characterized by the general opinion that it is a second order transition, see e.g. the standard textbooks [1, 2, 3]. This opinion is based on non-perturbative methods like lattice calculations

*e-mail: Michael.Bordag@itp.uni-leipzig.de

†e-mail: Skalozub@ff.dsu.dp.ua

and the average action resp. flow equation method [4, 5], however still lacks a formal, final proof.

In the perturbative approach numerous authors found a first order phase transition. Here the problem is in the infrared divergencies appearing there. A resummation of the relevant diagrams is necessary. This was known already since the pioneering papers in the field, e.g., [6]. The general outcome is that the perturbative methods do not give reliable results near the phase transition for two reasons:

1. Unwanted imaginary parts, for instance in the nontrivial minimum of the free energy (resp. the effective potential) appear indicating a instability or insufficient resummation. So for example, in the paper [7] a first order transition had been found on the level of super daisy graphs resummed. But then the authors included higher loop graphs and found an unwanted imaginary part.
2. Near the phase transition, the effective expansion parameter of the resummed perturbative series may become of order one making the contributions of higher order graphs unpredictable. For instance, it cannot be excluded that their inclusion may change the order of the phase transition.

In the present paper we consider the scalar ϕ^4 -theory in (3+1) dimensions in the perturbative approach. We perform a resummation of the perturbative series using the second Legendre transform and go beyond the super daisy resummation. We refine our previous results [8] obtained by a simpler resummation method which is restricted to the level of including the super daisy diagrams only. The disappearance of all unwanted imaginary parts was shown there and, thus, is repeated in Sec. 3. We note that this result implies that the imaginary part found in [7] from higher loop graphs is questionable.

Below, in Sec. 4, we calculate a number of contributions beyond the super daisy level. We show that, near the phase transition, they are, as expected, of the same order as the leading contributions. They turn out not to change the order of the phase transition but to affect its quantitative characteristics only making the transition a bit stronger first order.

To explain the controversy results on the order of the phase transition one could argue that the higher loop graphs not included in the perturbative approach might sum up to contributions that weaken the first order character until it turns into a second order one when all graphs are summed up finally. Our result, however, points in the opposite direction.

As technical tools we use the imaginary time formalism and functional methods. The most important tool is the use of the second Legendre transform¹ together with the necessary condition for an extremum of the free energy.

¹Is known since the beginning of the sixties ([9, 10] and can be found in textbooks [11], see also [12]).

We consider the scalar ϕ^4 -theory in (3+1) dimensions with the 'wrong sign' in the mass term which takes after symmetry breaking by shifting the field $\phi(x) \rightarrow v + \phi(x)$ the form

$$S[\phi] = \frac{m^2}{2}v^2 - \frac{\lambda}{4}v^4 + \int dx \left(\frac{1}{2}\phi(x)\mathbf{K}_\mu\phi(x) - \lambda v\phi(x)^3 - \frac{\lambda}{4}\phi(x)^4 + v(m^2 - \lambda v^2)\phi(x) \right) \quad (1)$$

with $\mathbf{K}_\mu = \square - \mu^2$ and the tree mass $\mu^2 = -m^2 + 3\lambda v^2$. The line Δ (free propagator) in the corresponding graphs is the inverse of \mathbf{K} : $\Delta = -\mathbf{K}^{-1}$. The quantity to be calculated is the free energy F . By means of $W = -TF$ it is related to the vacuum Green functions

$$Z = \int D\phi e^S, \quad (2)$$

where T is the temperature and its connected part reads

$$W = \log Z. \quad (3)$$

After performing the second Legendre transform the representation

$$W = S[0] + \frac{1}{2}Tr \log \beta - \frac{1}{2}Tr \Delta^{-1}\beta + W_2[\beta] \quad (4)$$

emerges, where all graphs are to be taken with the line β (instead of Δ) which is subject to the Schwinger-Dyson (SD) equation

$$\beta^{-1}(p) = \Delta^{-1} - \Sigma[\beta](p), \quad (5)$$

where

$$\Sigma[\beta](p) = 2 \frac{\delta W_2}{\delta \beta(p)} \quad (6)$$

are the self energy graphs with no propagator insertions, see, e.g., [12] for details. $W_2[\beta]$ is the sum of all two particle irreducible (2PI) graphs.

In the imaginary time formalism the operation 'Tr' is given by

$$Tr_p = T \sum_{l=-\infty}^{\infty} \int \frac{d^3\vec{p}}{(2\pi)^3} \quad (7)$$

with the momentum is $p = (2\pi Tl, \vec{p})$. We indicate the dependence on a functional argument β by square brackets and on the momentum p by round ones ($\beta(p)$ is the corresponding Fourier transform), δ is the variational derivative.

Turning to perturbation theory it is useful to start from W_2 . The first contribution to its perturbative expansion is the graph looking like a 'eight',

$$W_2^{(1)} = \frac{1}{8} \text{ (diagram of two circles connected by a vertical line) }. \quad (8)$$

By means of the formulas given above it generates the free energy with all super daisy diagrams summed up. In this way the results of our previous paper [8] are reproduced. In the present paper we go beyond the super daisy level by including the following (higher loop) graphs

$$W_2^{(h)} = \frac{1}{8} \text{ (circle with a dot)} + \frac{1}{12} \text{ (circle)} + \frac{1}{48} \text{ (circle with two dots)} + \sum_{n \geq 3} \frac{1}{2^n} \text{ (circle with } n \text{ dots)} + \sum_{n \geq 3} \frac{1}{2^{n+1}n} \text{ (circle with } n \text{ dots and a line)}, \quad (9)$$

where the dots symbolize further subgraphs to be inserted so that the number of vertices is n . This selection of graphs is motivated by the $1/N$ expansion where it includes all leading and first non-leading contributions.

2 The SD equation and the extrema of the free energy

The problem to be considered with respect to the phase transition is whether the free energy exhibits a nontrivial (i.e., at $v \neq 0$) maximum and minimum for some $v > 0$ at finite T . The common approach is to solve the SD equation (5) in some approximation and to insert its solution into the free energy (4) with a subsequent investigation of its dependence on the condensate v . Instead, we suggest to consider the necessary condition for the extremum of the free energy

$$\frac{\partial W}{\partial v^2} = 0 \quad (10)$$

together with the Schwinger-Dyson equation (5). In this way one variable can be eliminated and we obtain a simpler expression. In fact there are more simplifications behind. Away from the extrema of the free energy there are tadpole graphs generated by the linear term in the action (1) which must be taken into account. They disappear once Eq. (10) holds.

We obtain from Eq. (10) using (4) and $\Delta^{-1} = -\mathbf{K}_\mu^{-1} = p^2 - m^2 + 3\lambda v^2$ the condition

$$\frac{\partial W}{\partial v^2} = \frac{m^2}{2} - \frac{\lambda}{2}v^2 - \frac{3}{2}\lambda Tr\beta + Tr\frac{\partial\beta}{\partial v^2} \left(\frac{1}{2}\beta^{-1} - \frac{1}{2}\mathbf{K}_\mu + \frac{\delta W_2}{\delta\beta} \right) + \frac{\partial W_2}{\partial v^2} = 0, \quad (11)$$

which by means of (5) reduces² to

$$\lambda v^2 = m^2 - 3\lambda\Sigma_0 + 2\frac{\partial W_2}{\partial v^2} \quad (12)$$

with the notation

$$\Sigma_0 = Tr\beta. \quad (13)$$

²This is related to the known fact that the SD equation (5) is a stationary point of W considered as a functional of β , see e.g. [13].

A representation of Σ_0 is given in the Appendix.

Now the SD equation (5) can be rewritten eliminating v using (12) to become

$$\beta^{-1} = p^2 + 2m^2 - 9\lambda\Sigma_0 + 6\frac{\partial W_2}{\partial v^2} - \Sigma[\beta](p). \quad (14)$$

It holds in the extrema of the free energy and allows directly to determine number and kind of the extrema.

Being interested in the infrared behavior at high temperature we make the ansatz

$$\beta^{-1}(p) = p^2 + M^2 \quad (15)$$

and obtain from Eq. (14) the gap equation in the extremum

$$M^2 = 2m^2 - 9\lambda\Sigma_0 + 6\frac{\partial W_2}{\partial v^2} - \Sigma[\beta](0), \quad (16)$$

where the ansatz (15) is assumed to be used for β in the right hand side.

To proceed we need now to make the approximation defined by Eq. (9) on the level of the 2PI functional. The analytic expression for $W_2^{(h)}$ corresponding to Eq. (9) can be obtained from the graphical representation by inserting the line β and the vertex factors $-6\lambda v$ resp. -6λ for the three resp. four vertex. It reads

$$W_2^{(h)} = -\frac{3}{4}\lambda\Sigma_0^2 + 3\lambda^2 v^2 \Gamma + \lambda^2 D, \quad (17)$$

where we introduced the notations

$$\Gamma = Tr_q \beta(q) \Sigma_1(q) \frac{1 - 6\lambda\Sigma_1(q)}{1 + 3\lambda\Sigma_1(q)} \quad (18)$$

and

$$D = \frac{3}{4} Tr_q \Sigma_1(q)^2 \left(1 + 6 \sum_{n \geq 1} \frac{(-3\lambda\Sigma_1(q))^n}{n+2} \right) \quad (19)$$

as well as

$$\Sigma_1(p) = Tr_q \beta(q) \beta(p+q) = \text{---} \bigcirc \text{---} \quad (20)$$

Note that the infinite set of diagrams in Eq. (9) can be reproduced by a formal expansion of (17) with respect to powers of λ .

From these formulas we obtain by means of (6) the self energy in the form

$$\Sigma[\beta](p) = -3\lambda\Sigma_0 + 6\lambda^2 v^2 \frac{\delta\Gamma}{\delta\beta(p)} + 2\lambda^2 \frac{\delta D}{\delta\beta(p)}, \quad (21)$$

where we used obvious relations like $\delta\beta(p)/\delta\beta(q) = \delta(p-q)$ and $\delta\Sigma_1[\beta](p)/\delta\beta(q) = 2\beta(p+q)$. With $\frac{\partial W_2^{(h)}}{\partial v^2} = 3\lambda^2 \Gamma$ and Eq. (6) we obtain the SD equation in the extremum

$$\beta^{-1} = p^2 + 2 \left(m^2 - 3\lambda\Sigma_0 + 6\lambda^2 \Gamma \right) \left(1 - 3\lambda \frac{\delta\Gamma}{\delta\beta(p)} \right) + 6\lambda^2 \Gamma - 2\lambda^2 \frac{\delta D}{\delta\beta(p)}, \quad (22)$$

and from (16) the gap equation

$$M^2 = 2\tilde{\Gamma} \left(m^2 - 3\lambda\Sigma_0 + 6\lambda^2\Gamma \right) \quad (23)$$

in the given approximation. Here we used the relation

$$6\lambda^2\Gamma - 2\lambda^2 \frac{\delta D}{\delta\beta(p)} = 6\lambda^2 Tr_q (\beta(q) - \beta(p+q)) \Sigma_1(q) \frac{1 - 6\lambda\Sigma_1(q)}{1 + 3\lambda\Sigma_1(q)} \quad (24)$$

showing that this contribution to Eq. (22) vanishes for $p = 0$. The quantity $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma} \equiv 1 - 3\lambda \frac{\delta\Gamma}{\delta\beta(0)} = \frac{1 - 6\lambda\Sigma_1(0)}{1 + 3\lambda\Sigma_1(0)} + 18\lambda Tr_q \beta(q)^2 \frac{3\lambda\Sigma_1(q)(2 + 3\lambda\Sigma_1(q))}{(1 + 3\lambda\Sigma_1(q))^2}, \quad (25)$$

where Γ (18) was used.

Note that Eq. (23) does not contain the condensate v so that it is an equation to determine the mass M in the extrema of the free energy as a function of λ , T and m .

3 Symmetry restoration in super daisy approximation

Now, in order to establish the connection to our previous results [8] and to make the subsequent calculations more transparent we restrict ourselves for a moment to the level of super daisy graphs summed up only. This is obtained by setting Γ and D in the formula of the preceding section equal to zero. So, with

$$W_2^{(1)} = \frac{1}{8} \bigcirc = -\frac{3}{4}\lambda\Sigma_0^2 \quad (26)$$

we obtain the gap equation in the extremum in the form

$$M^2 = 2m^2 - 6\lambda\Sigma_0 \quad (27)$$

(cf. Eq. (37) in [8]). For the propagator $\beta^{-1} = p^2 + M^2$ is exact because Σ_0 does not depend on the momentum p . Equation (12) for the condensate reduces to

$$\lambda v^2 = m^2 - 3\lambda\Sigma_0 \quad (28)$$

so that we can note

$$\lambda v^2 = \frac{M^2}{2} \quad (29)$$

in the given approximation.

The solution of equation (27) can be investigated graphically by drawing both sides of the equation as function of M in one plot. A detailed analysis had been given in [8] showing clearly the behavior expected for a first order phase transition. For instance, in a certain temperature region the existence of two solutions corresponding to a maximum and a minimum of the free energy was found. For small coupling λ , which corresponds to high temperatures of the phase transition $T \sim \lambda^{-1/2}$, the function Σ_0 in the gap equation can be expanded in powers of T/M , see Eq. (55) in the appendix for details. Keeping the relevant orders the gap equation turns into

$$M^2 = 2m^2 - 6\lambda \left(\frac{T^2}{12} - \frac{MT}{4\pi} + \frac{m^2}{8\pi^2} + \frac{M^2}{16\pi^2} \left(\log \frac{(4\pi T)^2}{2m^2} - 2\gamma \right) \right) \quad (30)$$

and has the solutions

$$M_{\substack{Min \\ Max}} = \frac{3\lambda T}{4\pi\delta_3} \pm \sqrt{\frac{1}{\delta_3} \sqrt{\left(\frac{3\lambda T}{4\pi} \right)^2 \frac{1}{\delta_3} + 2m^2\delta_1 - \frac{\lambda T^2}{2}}}, \quad (31)$$

where the notations

$$\delta_1 = 1 - \frac{3\lambda}{8\pi^2} \quad (32)$$

and

$$\delta_3 = 1 + \frac{3\lambda}{8\pi^2} \left(\log \frac{(4\pi T)^2}{2m^2} - 2\gamma \right). \quad (33)$$

are used. Here, M_{Max} resp. M_{Min} (and v by means of Eq. (29)) correspond to the maximum resp. to the minimum of the free energy at $v \neq 0$. The lower spinodal temperature T_- , i.e., the temperature at which the maximum appears at $v = 0$, follows from $M_{Max} = 0$ to be $T_- = \frac{2m}{\sqrt{\lambda}}\sqrt{\delta_1}$ and the upper spinodal temperature T_+ , i.e., the temperature where the minimum and the maximum merge and disappear for $T > T_+$ follows from (31) to be $T_+ = \frac{2m}{\sqrt{\lambda}}\sqrt{\frac{\delta_1}{1 - \frac{9\lambda}{8\pi^2\delta_3}}}$. For small λ , in the first nontrivial order, they are

$$T_- = \frac{2m}{\sqrt{\lambda}} \left(1 - \frac{3\lambda}{16\pi^2} \right), \quad T_+ = \frac{2m}{\sqrt{\lambda}} \left(1 + 2\frac{3\lambda}{16\pi^2} \right), \quad (34)$$

and their ratio

$$\frac{T_+}{T_-} = 1 + \frac{9\lambda}{16\pi^2} \quad (35)$$

is larger than one. We note that this does not depend on the logarithm of T coming in with δ_3 . From Eq. (31) it is easy to calculate the masses in T_- and in T_+ :

$$\begin{aligned} M_{Min|T=T_-} &= \frac{3\sqrt{\lambda}}{\pi}m, & M_{Max|T=T_-} &= 0, \\ M_{Min|T=T_+} &= M_{Max|T=T_+} = \frac{3\sqrt{\lambda}}{2\pi}m. \end{aligned} \quad (36)$$

The temperature T_c of the phase transition where the free energy is the same in the trivial (i.e., at $v = 0$) and in the nontrivial minimum can be calculated in the given approximation quite easy. For this one needs the gap equation (5) at $v = 0$, $\beta^{-1} = p^2 - m^2 + 3\lambda\Sigma_0$. Note, that the self energy in the given approximation

$$\Sigma(p) = 2\frac{\delta}{\delta\beta(p)}\left(-\frac{3}{4}\lambda\Sigma_0^2\right) = -3\lambda\Sigma_0 \quad (37)$$

is independent of p and the ansatz $\beta^{-1}(p) = p^2 + M_0^2$ is exact again, where M_0 is the mass as solution of the gap equation at $v = 0$. In this way the gap equation there reads $M_0 = -m^2 + 3\lambda\Sigma_0$. Taking the approximation (55) for Σ_0 we obtain in a way similar to Eq. (31) its solution: $M_0 = \frac{-3\lambda T}{8\pi\delta_3} + \sqrt{\left(\frac{3\lambda T}{8\pi\delta_3}\right)^2 - \frac{m^2\delta_1}{\delta_3} - \frac{\lambda T^2}{4\delta_3}}$.

Now T_c can be obtained from equating the free energy at $v = 0$ and in the minimum:

$$W_0^{(1)} = W_e^{(1)} \quad \text{with} \quad \begin{cases} W_0 &= W(v \rightarrow 0, M \rightarrow M_0) \\ W_e &= W(v \rightarrow \sqrt{M_{Min}/2\lambda}, M \rightarrow M_{Min}), \end{cases} \quad (38)$$

where W are the connected Green functions (4). We rewrite them using $\Delta^{-1}\beta = 1 - \beta\Sigma$ following from the Schwinger-Dyson equation (5) and drop the infinite constant $Tr1$ appearing thereby. Again, in the given approximation, (26), using (37) we obtain

$$W^{(1)} = -\frac{m^2}{2}v^2 + \frac{\lambda}{4}v^4 - \frac{3}{4}\lambda\Sigma_0^2 + \frac{1}{2}\text{Tr} \ln \beta. \quad (39)$$

Now, $W_0^{(1)}$ resp. $W_e^{(1)}$ are obtained by inserting the corresponding values of v and M . After that, Eq. (38) can be approximated in the lowest nontrivial order in λ (thereby, in $\frac{1}{2}\text{Tr} \ln \beta$ all orders of λ shown in (53) have to be taken). The temperature T_c is in between T_- and T_+ , closer to T_+ . With $T_c = T_+(1 - \lambda\tau)$ we obtain $\tau = 0.0028$ as numerical solution of the emerging higher order algebraic equation.

So, on this stage of resummation, i.e., after summation of all super daisy diagrams which is generated by taking for W_2 the representation $W^{(1)}$ (26), we see clearly a first order phase transition. In the nontrivial minimum the mass M_{Min} (31) is positive and of order $\sqrt{\lambda}$:

$$\frac{3}{2\pi}\sqrt{\lambda}m = M_{Min|T_-} \leq M_{Min} \leq \frac{3}{\pi}\sqrt{\lambda}m = M_{Min|T_+}. \quad (40)$$

This mass has to be inserted into all higher loop graphs which are not included in the given approximation. As the theory is Euclidean and the propagator is $\beta(p) = 1/(p^2 + M^2)$ a consequence is the absence of unwanted imaginary parts. In this way the first problem mentioned in the introduction is settled.

Consider now the second. A simple counting of the expansion parameter of the perturbative series after all super daisy diagrams are summed up goes as follows. Let V , C resp. L be the number of vertices, loops resp. line of a graph. The vertex factors are proportional to λ (ϕ^4 -vertices) resp. λv (ϕ^3 -vertices). By means of formulas (29) and (36) the condensate is of order one, $v \sim m$, so that all vertex factors are of order λ . The summation/integration measure associated each loop is $T \sum_{l=-\infty}^{\infty} \int d\vec{p}/(2\pi)^3$ (7) and the factors of the lines are $1/((2\pi lT)^2 + \vec{p}^2 + M^2)$. For high T ($\sim \lambda^{-1/2}$) (34) the graphs can be approximated by taking the zeroth Matsubara frequency ($l = 0$). Then, doing a rescaling $\vec{p} \rightarrow \vec{p}M$, a factor $A = \lambda^V T^C M^{3C-2L}$ appears in front of each graph which itself is then a number of order one. By means of $C = L - V + 1$ and $M \sim \sqrt{\lambda}$ we obtain $A \sim \lambda$ so that it ceases to depend on the order of the perturbative expansion. This is a loss of the smallness of the effective expansion parameter. Although we know that all graphs are finite and of order one, we do not have a control over the sum of them. For instance, it cannot be excluded that summing up further (or all) graphs the order of the phase transition may change. Thus, super daisy approximation is insufficient to solve this problem.

4 Beyond the super daisy resummation

To account for graphs beyond the super daisy approximation let us return to the Schwinger-Dyson equation in the extremum (22). Now, because in the r.h.s. there are contributions depending on p , the simple ansatz (15) is strictly speaking insufficient. However, being interested in the infrared behavior it is anyway meaningful to use it at least in order to get a qualitatively correct result. After that we arrive at the gap equation (23). Here, Γ (18) and $\tilde{\Gamma}$ (25) contain the higher loop graphs. In order to get an estimate of these quantities at high T we make the following approximations.

Consider first Γ at $\lambda = 0$. It is the simple 2-loop graph $\Gamma(\lambda = 0) = \bigoplus$. Taking the contribution resulting from the zeroth Matsubara frequency only, it becomes $\Gamma(\lambda = 0) \sim \gamma T^2$ with some γ which we do not calculate explicitly. In fact, γ may exhibit a logarithmic dependence on T which we ignore, however. After that we observe that $\Sigma_1(q)$, for large q , behaves logarithmically (its superficial divergence degree is zero). Thereafter we approximate it by the first term of its high T expansion (57). It is a constant. Hence, we obtain for Γ (18) in this approximation

$$\Gamma = \frac{1-2\epsilon}{1+\epsilon} Tr_q \beta(q) \Sigma_1(q) \sim \frac{1-2\epsilon}{1+\epsilon} \gamma T^2, \quad (41)$$

where we used (57) and introduced the notation $\epsilon \equiv \frac{3\lambda T}{8\pi M}$ so that $3\lambda \Sigma_1(0) \sim \epsilon$ holds.

By the same procedure we obtain for $\tilde{\Gamma}$ (25) the approximation

$$\tilde{\Gamma}(\epsilon) = \frac{1-2\epsilon}{1+\epsilon} + 18\lambda T r_q \beta(q)^2 \frac{\epsilon(2+\epsilon)}{(1+\epsilon)^2}. \quad (42)$$

Here we indicated the dependence of $\tilde{\Gamma}$ on ϵ explicitly. We note $Tr_q \beta(q)^2 = \Sigma_1(0)$ and (57) so that we obtain finally

$$\tilde{\Gamma}(\epsilon) = \frac{1-2\epsilon}{1+\epsilon} + \frac{6\epsilon^2(2+\epsilon)}{(1+\epsilon)^2}. \quad (43)$$

This is a positive quantity of order one (note that in between T_- and T_+ , using (34) and (40), we have $\frac{1}{2} \leq \epsilon \leq \frac{1}{4}$).

Now we consider the gap equation in the extremum (23) in the leading non-trivial approximation and obtain

$$\begin{aligned} M^2 = & 2\tilde{\Gamma}(\epsilon) \left(m^2 - 3\lambda \left(\frac{T^2}{12} - \frac{MT}{4\pi} + \frac{m^2}{8\pi^2} + \frac{M^2}{16\pi^2} \left(\log \frac{(4\pi T)^2}{2m^2} - 2\gamma \right) \right) \right. \\ & \left. + 6\lambda^2 \gamma T^2 \right). \end{aligned} \quad (44)$$

This equation contains additional entries as compared to (30). Due to the dependence on the mass via ϵ it is not longer a quadratic but a higher order algebraic equation. Nevertheless, it is useful to rewrite it in analogy to (31). We obtain

$$M_{Min}^{Max} = \frac{3\lambda T}{4\pi\delta_3} \frac{\tilde{\Gamma}(\epsilon)}{\delta_3} \pm \sqrt{\frac{\tilde{\Gamma}(\epsilon)}{\delta_3} \sqrt{\left(\frac{3\lambda T}{4\pi\delta_3} \right)^2 \frac{\tilde{\Gamma}(\epsilon)}{\delta_3} + 2m^2\delta_1 - \frac{\lambda T^2}{2}\delta_\gamma}}, \quad (45)$$

with $\delta_\gamma \equiv 1 - 24\lambda\gamma$. Now by the same arguments as above we obtain instead of (34) for the spinodal temperatures

$$\begin{aligned} T_- &= \frac{2m}{\sqrt{\lambda}} \sqrt{\frac{\delta_1}{\delta_\gamma}} = \frac{2m}{\sqrt{\lambda}} \left(1 - \frac{3\lambda}{16\pi^2} + 12\lambda\gamma \right), \\ T_+ &= \frac{2m}{\sqrt{\lambda}} \sqrt{\frac{\delta_1}{\delta_\gamma - \frac{9\lambda\tilde{\Gamma}(\epsilon)}{8\pi^2\delta_3}}} = \frac{2m}{\sqrt{\lambda}} \left(1 - \frac{3\lambda}{16\pi^2} + 12\lambda\gamma + \frac{9\lambda\tilde{\Gamma}(\epsilon)}{16\pi^2} \right). \end{aligned} \quad (46)$$

Both these temperatures depend on γ and T_+ depends in addition on $\tilde{\Gamma}$. Their ratio is

$$\frac{T_+}{T_-} = 1 + \frac{9\lambda}{16\pi^2} \tilde{\Gamma}(\epsilon). \quad (47)$$

It remains to get some information on ϵ . As it comes in from T_+ , we consider M_{Min} as give by equation (45) at $T = T_+$. Taking into account that we need M_{Min} in the leading order ($\sim \sqrt{\lambda}$) only, we obtain the equation

$$M = \frac{3\sqrt{\lambda}m}{2\pi} \tilde{\Gamma}(\epsilon)$$

(at T_+ the square root in rhs. of Eq. (45) is zero). In the same approximation we have $\epsilon = \frac{3\sqrt{\lambda m}}{4\pi M}$ and the equation can be rewritten as

$$\frac{1}{2\epsilon} = \tilde{\Gamma}(\epsilon). \quad (48)$$

This is a fourth order algebraic equation with a solution $\epsilon^* = 0.318$ (one of the other solutions is negative, two are complex). We note $\tilde{\Gamma}(\epsilon^*) = 1.57$. So the gap (47) between the two spinodal temperatures becomes larger by about 50%. The other quantities characterizing the phase transition can be calculated in a similar way at last numerically. It is clear that they become all changed by a certain amount keeping the qualitative features unchanged.

So we arrive at the conclusion that the higher loop graphs given by Eq. (9) in comparison to (8) do not change the order of the phase transition, making it a bit stronger first order. As a consequence, they do not change the problem with the expansion parameter discussed at the end of the preceding section.

5 Discussion

Let us first discuss the main features of our analysis and the results obtained. As it has been realized many years ago [6, 14] in field theory the usual perturbation theory in coupling constant is not applicable in the infrared region and its series must be summed up to obtain more reliable results. A powerful method of resummation is the second Legendre transform giving the possibility to express the perturbative expansion in terms of two particle irreducible diagrams. Therefore it accounts for the infrared divergencies coming from propagator insertions. In the present paper it has been applied to investigate the temperature phase transition in the scalar $\phi^4(x)$ theory with spontaneous symmetry breaking.

As an important step, the SD equation in the minimum of the free energy functional was used. This considerably simplifies computations making them more transparent and tractable. In order to analyze the phase transition we adopted the ansatz (15) for the full propagator which is natural at high temperature. As we have seen in Sect. 3, all daisy and super daisy diagrams can be summed up by taking the ‘eight ‘-diagram in W_2 , (26). In fact, this resummation renders the free energy in the extremum real solving the problem of eliminating the instability connected with the imaginary part which constitutes a necessary consistency condition. The type of the phase transition was found to be of first order. At the same time, as it also follows from the analysis in Sec. 3, near the phase transition the effective expansion parameter became of order one making the contribution of subsequent graphs questionable.

To go beyond the super daisy approximation we have included into consideration an infinite set of diagrams in the 2PI functional (9) and, as a consequence, the corresponding momentum dependent diagrams in the SD equation (22). Their

choice is motivated by the possibility to handle them. Besides, in an extension to the $O(N)$ -model these graphs would constitute the first non-leading order for large N . In Sec. 4 we calculated their contribution to the SD equation in the extremum of the free energy. Within this approximation they turned out not to change things essentially. Their role reduces to a redefinition of the parameters of the phase transition by a finite factor of order of $3/2$ making the transition a bit stronger first order. As a consequence, the estimation of the effective expansion parameter given at the end of Sec. 3 on the super daisy resummation level, remains valid on the higher loop resummation of Sec. 4 too. We come to the conclusion that even this resummation is still insufficient to get a reliable result from perturbation theory near the phase transition.

We believe, anyway, that the technical tools developed here, mainly the joining of the SD equation with the necessary condition for an extremum will find applications in gauge theories. There, as is known, the free energy is independent of the gauge fixing parameter only in its minima. Therefore, additional vertex resummations, such as for instance done in Ref. [15] in order to obtain a gauge invariant result, may be avoided eventually. The realization of these ideas in gauge theories is a problem left for the future.

Acknowledgment

VS thanks the Saxonian State Ministry for Science and Arts for support under the grant 4-7531.50-04-0361-00/12 and the University of Leipzig for kind hospitality.

Appendix

The function $V_1(M) \equiv \frac{1}{2} \text{Tr} \ln \beta$ appearing in (39) reads

$$V_1(M) = -\frac{1}{2} \text{Tr} \ln \Delta = T \sum_{l=-\infty}^{\infty} \int \frac{d\vec{k}}{(2\pi)^3} \ln \left((2\pi l T)^2 + \vec{k}^2 + M^2 \right). \quad (49)$$

Removing the ultraviolet divergencies the explicit expression becomes

$$V_1(M) = \frac{1}{64\pi^2} \left\{ 4m^2 M^2 + M^4 \left[\ln \frac{M^2}{2m^2} - \frac{3}{2} \right] \right\} - \frac{M^2 T^2}{2\pi^2} S_2 \left(\frac{M}{T} \right), \quad (50)$$

where the last term is the temperature dependent contribution. The function S_2 can be represented as a fast converging sum over the modified Bessel function K_2 and by an integral representation as well:

$$S_2(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(nx) = \frac{1}{3x^2} \int_x^{\infty} dn \frac{(n^2 - x^2)^{3/2}}{e^n - 1}. \quad (51)$$

We use

$$\frac{\partial V_1}{\partial V}|_{v=m/\sqrt{\lambda}} = 0 \quad \text{and} \quad \frac{\partial^2 V_1}{\partial v^2}|_{v=m/\sqrt{\lambda}} = 0$$

as normalization conditions at $T = 0$. These conditions are chosen in a way that the minimum of the free energy is at $T = 0$ the same as on the tree level. It is possible to do so as the counter-terms can be chosen at $T = 0$, see for example a recent new proof [16].

The expansion of $S_2(x)$ for small x reads

$$S_2(x) = \frac{\pi^4}{45x^2} - \frac{\pi^2}{12} + \frac{\pi x}{6} + \frac{x^2}{32} \left(2\gamma - \frac{3}{2} + 2 \ln \frac{x}{4\pi} \right) + O(x^3), \quad (52)$$

giving rise to the high temperature expansion of V_1 which is at once the expansion for small M :

$$\begin{aligned} V_1(M) = & \frac{-\pi^2 T^4}{90} + \frac{M^2 T^2}{24} - \frac{M^3 T}{12\pi} \\ & + \frac{1}{64\pi^2} \left\{ M^4 \left[\ln \frac{(4\pi T)^2}{2m^2} - 2\gamma \right] + 4m^2 M^2 \right\} + M^2 T^2 O\left(\frac{M}{T}\right). \end{aligned} \quad (53)$$

Here γ is the Euler constant.

The function Σ_0 defined in Eq. (13) can be obtained by differentiating V_1

$$\Sigma_0 = -\frac{\partial}{\partial M^2} V_1(M). \quad (54)$$

Its high temperature expansion is

$$\Sigma_0(M) = \frac{T^2}{12} - \frac{MT}{4\pi} + \frac{1}{16\pi^2} \left\{ M^2 \left(\ln \frac{(4\pi T)^2}{2m^2} - 2\gamma \right) + 2m^2 \right\} + MT O\left(\frac{M}{T}\right). \quad (55)$$

Similar we obtain for $\Sigma_1(p)$ (20) at $p = 0$

$$\Sigma_1(0) = -\frac{\partial}{\partial M^2} \Sigma_0 \quad (56)$$

with the high temperature expansion

$$\Sigma_1(0) = \frac{T}{4\pi M} + O(1). \quad (57)$$

References

- [1] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. Clarendon Press, 1996.

- [2] A. Linde. *Particle Physics and Inflationary Cosmology*. harwood academic publishers, 1990.
- [3] J.I. Kapusta. *Finite-temperature field theory*. Cambridge University Press, 1989.
- [4] N. Tetradis and C. Wetterich. The high temperature phase transition for ϕ^4 theories. *Nucl. Phys.*, B398:659–696, 1993.
- [5] M. Reuter, N. Tetradis, and C. Wetterich. The large n limit and the high temperature phase transition for the ϕ^{**4} theory. *Nucl. Phys.*, B401:567–590, 1993.
- [6] L. Dolan and R. Jackiw. Symmetry behavior at finite temperature. *Phys. Rev.*, D9:3320–3341, 1974.
- [7] J. R. Espinosa, M. Quiros, and F. Zwirner. On the phase transition in the scalar theory. *Phys. Lett.*, B291:115–124, 1992.
- [8] M. Bordag and V. Skalozub. On symmetry restoration at finite temperature (scalar case). 1999. hep-th/9908003.
- [9] J. M. Luttinger and J. C. Ward. Ground-state energy of a many-fermion system. ii. *Phys. Rev.*, 118:1417, 1960.
- [10] G. Baym. Self-consistent approximations in many-body system. *Phys. Rev.*, 127:1391, 1962.
- [11] A.N. Vasiliev. *Functional Methods in Quantum Field Theory and Statistical Physics*. Gordon and Breach, 1998. (А.Н. Васильев, Функциональные методы в квантовой теории поля и статистике, издательство Ленинградского университета, 1976) (in Russian).
- [12] John M. Cornwall, R. Jackiw, and E. Tomboulis. Effective action for composite operators. *Phys. Rev.*, D8:2428–2445, 1974.
- [13] J. P. Blaizot, E. Iancu, and A. Rebhan. The entropy of the qcd plasma. *Phys. Rev. Lett.*, 83:2906–2909, 1999.
- [14] A. D. Linde. Phase transitions in gauge theories and cosmology. *Rept. Prog. Phys.*, 42:389, 1979.
- [15] W. Buchmuller and O. Philipsen. Abelian versus nonabelian higgs model in three-dimensions. *Phys. Lett.*, B354:403–408, 1995.
- [16] Christoph Kopper, Volkhard F. Muller, and Thomas Reisz. Temperature independent renormalization of finite temperature field theory. 2000. hep-th/0003254.